

Exercise 2

Derive the integration formula

$$\int_0^{\infty} \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}$$

by integrating the function

$$f(z) = \frac{z^{-1/2}}{z^2+1} = \frac{e^{(-1/2)\log z}}{z^2+1} \quad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2} \right)$$

over the indented contour appearing in Fig. 109 (Sec. 90).

Solution

In order to evaluate this integral, consider the given function in the complex plane $f(z)$ and the contour in Fig. 109. Singularities occur where the denominator is equal to zero.

$$\begin{aligned} z^2 + 1 &= 0 \\ z &= \pm i \end{aligned}$$

The singular point of interest to us is the one that lies within the closed contour, $z = i$. Since $z^{-1/2}$ can be written in terms of $\log z$, a branch cut for the function needs to be chosen.

$$z^{-1/2} = \exp\left(-\frac{1}{2}\log z\right)$$

We choose it to be the axis of negative imaginary numbers.

$$= \exp\left[-\frac{1}{2}(\ln r + i\theta)\right], \quad \left(|z| > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2} \right),$$

where $r = |z|$ is the magnitude of z and $\theta = \arg z$ is the argument of z .

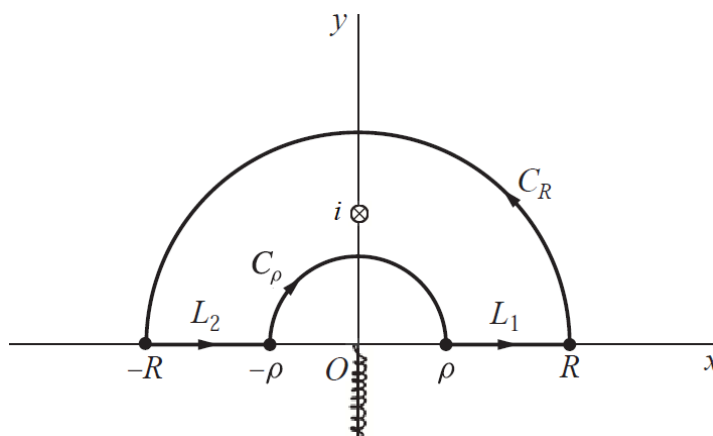


Figure 1: This is Fig. 109 with the singularity at $z = i$ marked. The squiggly line represents the branch cut ($|z| > 0, -\pi/2 < \theta < 3\pi/2$).

According to Cauchy's residue theorem, the integral of $z^{-1/2}/(z^2 + 1)$ around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{z^{-1/2}}{z^2 + 1} dz = 2\pi i \operatorname{Res}_{z=i} \frac{z^{-1/2}}{z^2 + 1}$$

This closed loop integral is the sum of four integrals, one over each arc in the loop.

$$\int_{L_1} \frac{z^{-1/2}}{z^2 + 1} dz + \int_{L_2} \frac{z^{-1/2}}{z^2 + 1} dz + \int_{C_\rho} \frac{z^{-1/2}}{z^2 + 1} dz + \int_{C_R} \frac{z^{-1/2}}{z^2 + 1} dz = 2\pi i \operatorname{Res}_{z=i} \frac{z^{-1/2}}{z^2 + 1}$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} L_1 : z &= re^{i0}, & r = \rho &\rightarrow r = R \\ L_2 : z &= re^{i\pi}, & r = R &\rightarrow r = \rho \\ C_\rho : z &= \rho e^{i\theta}, & \theta = \pi &\rightarrow \theta = 0 \\ C_R : z &= Re^{i\theta}, & \theta = 0 &\rightarrow \theta = \pi \end{aligned}$$

As a result,

$$\begin{aligned} 2\pi i \operatorname{Res}_{z=i} \frac{z^{-1/2}}{z^2 + 1} &= \int_\rho^R \frac{(re^{i0})^{-1/2}}{(re^{i0})^2 + 1} (dr e^{i0}) + \int_R^\rho \frac{(re^{i\pi})^{-1/2}}{(re^{i\pi})^2 + 1} (dr e^{i\pi}) + \int_{C_\rho} \frac{z^{-1/2}}{z^2 + 1} dz + \int_{C_R} \frac{z^{-1/2}}{z^2 + 1} dz \\ &= \int_\rho^R \frac{r^{-1/2}}{r^2 + 1} dr + \int_R^\rho \frac{r^{-1/2} e^{-i\pi/2}}{(-r)^2 + 1} (-dr) + \int_{C_\rho} \frac{z^{-1/2}}{z^2 + 1} dz + \int_{C_R} \frac{z^{-1/2}}{z^2 + 1} dz \\ &= \int_\rho^R \frac{r^{-1/2}}{r^2 + 1} dr + \int_\rho^R \frac{r^{-1/2} e^{-i\pi/2}}{r^2 + 1} dr + \int_{C_\rho} \frac{z^{-1/2}}{z^2 + 1} dz + \int_{C_R} \frac{z^{-1/2}}{z^2 + 1} dz \\ &= (1 + e^{-i\pi/2}) \int_\rho^R \frac{r^{-1/2}}{r^2 + 1} dr + \int_{C_\rho} \frac{z^{-1/2}}{z^2 + 1} dz + \int_{C_R} \frac{z^{-1/2}}{z^2 + 1} dz. \end{aligned}$$

Take the limit now as $\rho \rightarrow 0$ and $R \rightarrow \infty$. The integral over C_ρ tends to zero, and the integral over C_R tends to zero. Proof for these statements will be given at the end.

$$(1 + e^{-i\pi/2}) \int_0^\infty \frac{r^{-1/2}}{r^2 + 1} dr = 2\pi i \operatorname{Res}_{z=i} \frac{z^{-1/2}}{z^2 + 1}$$

Writing the denominator as $z^2 + 1 = (z + i)(z - i)$, we see that the multiplicity of the $z - i$ factor is 1. Consequently, the residue at $z = i$ can be calculated by

$$\operatorname{Res}_{z=i} \frac{z^{-1/2}}{z^2 + 1} = \phi(i),$$

where $\phi(z)$ is the same function as $f(z)$ without the $z - i$ factor.

$$\phi(z) = \frac{z^{-1/2}}{z + i} \Rightarrow \phi(i) = \frac{i^{-1/2}}{2i} = \frac{(e^{i\pi/2})^{-1/2}}{2i} = \frac{e^{-i\pi/4}}{2i}$$

So then

$$\operatorname{Res}_{z=i} \frac{z^{-1/2}}{z^2 + 1} = \frac{e^{-i\pi/4}}{2i}$$

and

$$\begin{aligned}(1 + e^{-i\pi/2}) \int_0^\infty \frac{r^{-1/2}}{r^2 + 1} dr &= 2\pi i \left(\frac{e^{-i\pi/4}}{2i} \right) \\ &= \pi e^{-i\pi/4}.\end{aligned}$$

Divide both sides by $1 + e^{-i\pi/2}$.

$$\begin{aligned}\int_0^\infty \frac{r^{-1/2}}{r^2 + 1} dr &= \pi \frac{e^{-i\pi/4}}{1 + e^{-i\pi/2}} \\ &= \pi \frac{1}{e^{i\pi/4} + e^{-i\pi/4}} \\ &= \pi \frac{1}{2 \cos(\pi/4)} \\ &= \pi \frac{1}{\sqrt{2}}\end{aligned}$$

Therefore, changing the dummy integration variable to x ,

$$\boxed{\int_0^\infty \frac{1}{\sqrt{x}(x^2 + 1)} dx = \frac{\pi}{\sqrt{2}}.}$$

The Integral Over C_ρ

Our aim here is to show that the integral over C_ρ tends to zero in the limit as $\rho \rightarrow 0$. The parameterization of the small semicircular arc in Figure 1 is $z = \rho e^{i\theta}$, where θ goes from π to 0.

$$\begin{aligned}\int_{C_\rho} \frac{z^{-1/2}}{z^2 + 1} dz &= \int_\pi^0 \frac{(\rho e^{i\theta})^{-1/2}}{(\rho e^{i\theta})^2 + 1} (\rho i e^{i\theta} d\theta) \\ &= \int_\pi^0 \frac{\rho^{1/2} e^{i\theta/2}}{\rho^2 e^{2i\theta} + 1} (i d\theta)\end{aligned}$$

Take the limit of both sides as $\rho \rightarrow 0$.

$$\lim_{\rho \rightarrow 0} \int_{C_\rho} \frac{z^{-1/2}}{z^2 + 1} dz = \lim_{\rho \rightarrow 0} \int_\pi^0 \frac{\rho^{1/2} e^{i\theta/2}}{\rho^2 e^{2i\theta} + 1} (i d\theta)$$

The limits of integration are constant, so the limit may be brought inside the integral.

$$\lim_{\rho \rightarrow 0} \int_{C_\rho} \frac{z^{-1/2}}{z^2 + 1} dz = \int_\pi^0 \lim_{\rho \rightarrow 0} \frac{\rho^{1/2} e^{i\theta/2}}{\rho^2 e^{2i\theta} + 1} (i d\theta)$$

Because of $\rho^{1/2}$ in the numerator, the limit is zero. Therefore,

$$\lim_{\rho \rightarrow 0} \int_{C_\rho} \frac{z^{-1/2}}{z^2 + 1} dz = 0.$$

The Integral Over C_R

Our aim here is to show that the integral over C_R tends to zero in the limit as $R \rightarrow \infty$. The parameterization of the large semicircular arc in Figure 1 is $z = R e^{i\theta}$, where θ goes from 0 to π .

$$\begin{aligned}\int_{C_R} \frac{z^{-1/2}}{z^2 + 1} dz &= \int_0^\pi \frac{(R e^{i\theta})^{-1/2}}{(R e^{i\theta})^2 + 1} (R i e^{i\theta} d\theta) \\ &= \int_0^\pi \frac{R^{1/2} e^{i\theta/2}}{R^2 e^{2i\theta} + 1} (i d\theta) \\ &= \int_0^\pi \frac{R^{1/2} e^{i\theta/2}}{R^2 (e^{2i\theta} + \frac{1}{R^2})} (i d\theta) \\ &= \int_0^\pi \frac{1}{R^{3/2}} \frac{e^{i\theta/2}}{(e^{2i\theta} + \frac{1}{R^2})} (i d\theta)\end{aligned}$$

Take the limit of both sides as $R \rightarrow \infty$. Since the limits of integration are constant, the limit may be brought inside the integral.

$$\lim_{R \rightarrow \infty} \int_{C_R} \frac{z^{-1/2}}{z^2 + 1} dz = \int_0^\pi \lim_{R \rightarrow \infty} \frac{1}{R^{3/2}} \frac{e^{i\theta/2}}{(e^{2i\theta} + \frac{1}{R^2})} (i d\theta)$$

Because of $R^{3/2}$ in the denominator, the limit is zero. Therefore,

$$\lim_{R \rightarrow \infty} \int_{C_R} \frac{z^{-1/2}}{z^2 + 1} dz = 0.$$